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Dynamical symmetries and conserved quantities

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Abstract. The invariance properties of second order dynamical systems under velocity dependent transformations of the coordinates and time are studied. For Lagrangian systems the connection between Noether conserved quantities and dynamical symmetries is not too direct; however, we show that for general systems dynamical symmetries always possess associated conserved quantities, which are invariants of the symmetry group itself. In the special case of point symmetries this yields the result that the associated conserved quantity is an invariant of the first extended group.

1. Introduction

It is well known from Noether's theorem (Noether 1918) that there is a close connection between the conserved quantities of a Lagrangian dynamical system and those transformations of coordinates and time which preserve the action integral. Transformations which leave the action invariant form a Lie group, which itself may be a proper subgroup of the group of transformations leaving the equations of motion invariant (Lutzky 1978). We shall refer to transformations of the coordinates and time which leave the equations of motion invariant as point symmetries.

It is also of interest to study the invariance of dynamical systems under more general transformations, which allow the new coordinates and time to depend on the old velocities, as well as on the old coordinates and time. Such transformations operate on trajectories in space–time, and will be called dynamical symmetries if they preserve the equations of motion. The invariance properties of differential equations under derivative-dependent transformations have been studied in different contexts by several authors (Ovsjannikov 1962, Anderson *et al* 1972a, 1972b, 1972c); in particular, Anderson and Davison (1974) have given a treatment which may be applied to second-order dynamical systems. Their methods constitute an extension of the Ovsjannikov theory, which is itself a generalisation of the Lie theory of extended groups (Cohen 1931). However, they do not make use of the Lagrangian formalism or Noether's theorem. A recent investigation (Lutzky 1978) shows how a treatment using Noether's theorem relates to and complements the extended group treatment for the case of point symmetries. This relationship is further explored here for the case of dynamical symmetries, and it is shown that some new features appear. We shall find that in the case of Lagrangian systems the connection between conserved quantities and dynamical symmetries is not too direct; for instance, there always exists an *infinite* number of linearly independent dynamical symmetries which have no associated Noether constant of the motion. However, we will establish a connection between dynamical symmetries and conserved quantities for general (that is not necessarily

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Lagrangian) dynamical systems; it will turn out that the conserved quantities associated with a given dynamical symmetry are invariants of the symmetry group itself. In addition we derive a condition for invariance transformations in a different manner than that employed by Anderson and Davison (1974); the criterion used here will be that conserved quantities must be transformed into conserved quantities. Finally we demonstrate that our results apply also to the special case of point symmetries; in particular, we show that the conserved quantities associated with a point symmetry group are invariants of the first extended group. This has been previously proved only for the special case of a Noether invariant associated with a point symmetry of a one-dimensional Lagrangian system (Lutzky 1978).

We call any transformation which possesses an associated Noether conserved quantity a Noether transformation. It follows from our results that a symmetry transformation need not be a Noether transformation, and a Noether transformation need not be a symmetry transformation. This has previously been noted by Rosen (1972) in the context of classical field theory; however, his point of view and approach are quite different from ours.

2. Dynamical symmetries

We briefly describe here some properties of dynamical symmetries. Consider a transformation from the $p+1$ variables $q_1, q_2, \ldots, q_p, t$ to the $p+1$ variables $Q_1, Q_2, \ldots, Q_p, T$, and let us suppose that the new variables also depend on the p time derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_p$. This transformation may be written

$$Q_l = Q_l(q, \dot{q}, t) \qquad T = T(q, \dot{q}, t) \tag{1}$$

where the q_l are to be considered functions of the time, t. The equations $q_l = q_l(t)$ define a curve in q, t space, with $\dot{q}_l = \mathrm{d}q_l/\mathrm{d}t$; the above transformation may therefore be said to act on curves in q, t space, and to produce curves in Q, T space. Expressing q_l, $\dot{q}_l$ in terms of t and eliminating t from (1) yields the resulting curve $Q_l = Q_l(T)$ in Q, T space. The time derivatives $\dot{Q}_l = \mathrm{d}Q_l/\mathrm{d}T$ may be found in terms of q_l, $\dot{q}_l$, t as follows:

$$\dot{Q}_l = \frac{\{\mathrm{d}Q_l/\mathrm{d}t\}}{\{\mathrm{d}T/\mathrm{d}t\}} = \frac{\{\partial Q_l/\partial t + (\partial Q_l/\partial q_m)\dot{q}_m + (\partial Q_l/\partial \dot{q}_m)\ddot{q}_m\}}{\{\partial T/\partial t + (\partial T/\partial q_m)\dot{q}_m + (\partial T/\partial \dot{q}_m)\ddot{q}_m\}}. \tag{2}$$

If we assume that the curve $q_l = q_l(t)$ satisfies the differential equation

$$\ddot{q}_l = \alpha_l(q, \dot{q}, t) \qquad (l = 1, 2, \ldots, p) \tag{3}$$

then equation (2) may be written

$$\dot{Q}_l = \frac{\{\partial Q_l/\partial t + (\partial Q_l/\partial q_m)\dot{q}_m + (\partial Q_l/\partial \dot{q}_m)\alpha_m\}}{\{\partial T/\partial t + (\partial T/\partial q_m)\dot{q}_m + (\partial T/\partial \dot{q}_m)\alpha_m\}}. \tag{4}$$

Equations (1) and (4) constitute a transformation from the $q, \dot{q}, t$ variables to the $Q, \dot{Q}, T$ variables; this transformation has the special property of mapping solution curves of $\ddot{q}_l = \alpha_l(q, \dot{q}, t)$ into curves $Q_l = Q_l(T)$ in Q, T space. If this transformation is to be a symmetry of (3), it must transform solution curves into solution curves; that is, $Q_l = Q_l(T)$ must be a solution of $\ddot{Q}_l = \alpha_l(Q, \dot{Q}, T)$. We now derive a criterion for this to be the case. Our method will be to assume that the transformation is associated with a Lie group, and we shall obtain a condition on the infinitesimal generator of the group.

Let us suppose that the equations (1) contain a parameter θ, with respect to which they form a Lie group. To first order in θ we may write

$$Q_l = q_l + \theta\eta_l(q, \dot{q}, t) \qquad T = t + \theta\xi(q, \dot{q}, t). \tag{5}$$

Differentiating, we may form

$$\partial Q_l/\partial t = \theta\ \partial\eta_l/\partial t, \qquad \partial Q_l/\partial q_m = \delta_{lm} + \theta\ \partial\eta_l/\partial q_m, \tag{6}$$

with similar expressions for the other derivatives appearing in (4). Using these results in (4) and expanding to first order in θ we obtain

$$\dot{Q}_l = \dot{q}_l + \theta[\dot{\eta}_l - \dot{\xi}\dot{q}_l], \tag{7}$$

where

$$\dot{\eta}_l = \partial\eta_l/\partial t + (\partial\eta_l/\partial q_m)\dot{q}_m + (\partial\eta_l/\partial\dot{q}_m)\alpha_m, \qquad \dot{\xi} = \partial\xi/\partial t + (\partial\xi/\partial q_m)\dot{q}_m + (\partial\xi/\partial\dot{q}_m)\alpha_m. \tag{8}$$

From (5) and (7) we see that the infinitesimal generator of the group may be written in the form

$$E = \xi\ \partial/\partial t + \eta_l\ \partial/\partial q_l + [\dot{\eta}_l - \dot{\xi}\dot{q}_l]\ \partial/\partial\dot{q}_l; \tag{9}$$

the finite equations of transformation are

$$Q_l = \exp(\theta E)\ q_l \qquad T = \exp(\theta E)\ t \qquad \dot{Q}_l = \exp(\theta E)\ \dot{q}_l. \tag{10}$$

The special form of the coefficient of $\partial/\partial\dot{q}_l$ in equation (9) guarantees that when the group acts upon a solution curve of equation (3), the result constitutes a curve in Q, T space, with the $\dot{Q}_l$ given by (10c) being precisely the time derivatives along the curve. We now derive conditions on $\xi(q, \dot{q}, t)$ and $\eta_l(q, \dot{q}, t)$ which ensure that the resultant curve in Q, T space is itself a solution of $\ddot{Q}_l = \alpha_l(Q, \dot{Q}, T)$.

Let us suppose that the transformation (10) permutes solutions of equation (3) among themselves, and let $\phi(q, \dot{q}, t)$ be a conserved quantity for (3). Then we may put

$$\exp(\theta E)\ (\phi(q, \dot{q}, t)) = \phi(\exp(\theta E)\ q, \exp(\theta E)\ \dot{q}, \exp(\theta E)\ t) = \phi(Q, \dot{Q}, T) = \Psi(\theta, q, \dot{q}, t).$$

If $\phi(q, \dot{q}, t) = C_1$ for the solution $q_l = q_l(t)$, then $\phi(Q, \dot{Q}, T) = C_2$ for the solution $Q_l = Q_l(T)$; it then follows that $\Psi(\theta, q, \dot{q}, t)$ is also a conserved quantity. Furthermore, in the expansion of Ψ in powers of θ, each coefficient must itself be a conserved quantity. In particular, consideration of first order terms shows that $E\{\phi\}$ is a conserved quantity if ϕ is. Expressing this result in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E\{\phi\} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\xi\frac{\partial\phi}{\partial t} + \eta_l\frac{\partial\phi}{\partial q_l} + [\dot{\eta}_l - \dot{\xi}\dot{q}_l]\frac{\partial\phi}{\partial\dot{q}_l}\right) = 0 \tag{11}$$

allows us to derive a condition which $\xi(q, \dot{q}, t)$ and $\eta_l(q, \dot{q}, t)$ must satisfy in order for (10) to generate a symmetry transformation of (3). This is done by explicitly carrying out the total time differentiation in (11), and simplifying the resultant expression by use of the following relations:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial q_l}\dot{q}_l + \frac{\partial\phi}{\partial\dot{q}_l}\alpha_l = 0 \tag{11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi}{\partial\dot{q}_l}\right) = -\frac{\partial\phi}{\partial q_l} - \frac{\partial\phi}{\partial\dot{q}_m}\frac{\partial\alpha_m}{\partial\dot{q}_l} \tag{11b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi}{\partial q_l}\right) = -\frac{\partial\phi}{\partial\dot{q}_m}\frac{\partial\alpha_m}{\partial q_l} \tag{11c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi}{\partial t}\right) = -\frac{\partial\phi}{\partial\dot{q}_m}\frac{\partial\alpha_m}{\partial t}. \tag{11d}$$

Equation ($11a$) is simply the statement that ϕ is a conserved quantity. To derive ($11b$) we put

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi}{\partial\dot{q}_l}\right) = \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial\dot{q}_l}\right) + \frac{\partial}{\partial q_m}\left(\frac{\partial\phi}{\partial\dot{q}_l}\right)\dot{q}_m + \frac{\partial}{\partial\dot{q}_m}\left(\frac{\partial\phi}{\partial\dot{q}_l}\right)\alpha_m$$

$$= \frac{\partial}{\partial\dot{q}_l}\left[\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial q_m}\dot{q}_m + \frac{\partial\phi}{\partial\dot{q}_m}\alpha_m\right] - \frac{\partial\phi}{\partial q_l} - \frac{\partial\phi}{\partial\dot{q}_m}\frac{\partial\alpha_m}{\partial\dot{q}_l}.$$

Because of ($11a$) the quantity in rectangular brackets vanishes, yielding ($11b$). Equations ($11c$) and ($11d$) may be obtained in a similar manner, and using ($11a, b, c, d$) in the expanded form of (11) then yields

$$(\mathrm{d}/\mathrm{d}t)E\{\phi\} = [\ddot{\eta}_l - \dot{q}_l\ddot{\xi} - 2\dot{\xi}\alpha_l - E\{\alpha_l\}](\partial\phi/\partial\dot{q}_l) = 0.$$

Since this must hold for any conserved quantity, we obtain the conditions

$$\ddot{\eta}_l - \dot{q}_l\ddot{\xi} - 2\dot{\xi}\alpha_l - E\{\alpha_l\} = 0 \qquad (l = 1, 2, \ldots, p) \tag{12}$$

which constitute a set of p equations in the $p+1$ quantities

$$\xi(q, \dot{q}, t),\ \eta_1(q, \dot{q}, t), \ldots, \eta_p(q, \dot{q}, t).$$

This criterion is expressed in a more compact form than that given by Anderson and Davison (1974), and its derivation does not require the use of the general Ovsjannikov theory.

We note here that the transformation (10), with ξ and η_l subject to (12), can always be interpreted as being simply a point transformation of the $2p+1$ independent variables q_l, $\dot{q}_l$, t. In this case, the domain of the operators $\exp(\theta E)$ is the manifold of all possible values of the variables. However, if the transformation is considered to act on trajectories and to produce trajectories, then it is clear from what has gone on before that both the domain and the range of the operators are limited to the manifold of solutions of (3).

As an example, we may consider the one-dimensional harmonic oscillator, whose equation of motion, in suitably normalised coordinates, may be written

$$\ddot{q} + q = 0 \tag{13}$$

so that

$$\alpha(q, \dot{q}, t) = -q. \tag{14}$$

Then the condition (12) becomes

$$\ddot{\eta} + \eta - \dot{q}\ddot{\xi} + 2q\dot{\xi} = 0. \tag{15}$$

If we assume the forms $\xi = \dot{q}\sigma(q, t)$, $\eta = \eta(q, t)$, then (15) has the solutions

$$\xi = q\dot{q} \qquad \eta = -q^3 \tag{16a}$$

$$\xi = \dot{q}\sin t \qquad \eta = -q^2\sin t \tag{16b}$$

$$\xi = \dot{q}\cos t \qquad \eta = -q^2 \cos t \tag{16c}$$

$$\xi = 0 \qquad \eta = q \tag{16d}$$

$$\xi = 0 \qquad \eta = \sin t \tag{16e}$$

$$\xi = 0 \qquad \eta = \cos t. \tag{16f}$$

These symmetry transformations for the harmonic oscillator have been found previously by Anderson and Davison (1974). We will later consider (16a) in more detail; the infinitesimal generator of this group is

$$E = q\dot{q}(\partial/\partial t) - q^3(\partial/\partial q) - (2q^2 + \dot{q}^2)\dot{q}(\partial/\partial \dot{q}). \tag{17}$$

3. Dynamical symmetries and Lagrangian systems

We now study the connection between dynamical symmetries and conserved quantities, limiting ourselves at first to systems derivable from a Lagrangian $L(q, \dot{q}, t)$. In analogy with a previous treatment of point symmetries (Lutzky 1978) we write the general Noether-type conserved quantity in the form

$$\Phi = (\xi\dot{q}_l - \eta_l)\, \partial L/\partial \dot{q}_l - \xi L + f \tag{18}$$

where

$$\xi = \xi(q, \dot{q}, t) \qquad \eta_l = \eta_l(q, \dot{q}, t) \qquad f = f(q, t).$$

We ask the following questions. (i) Given a conserved quantity of the form (18) under what circumstances do $\xi, \eta_1, \eta_2, \ldots, \eta_p$ determine a dynamical symmetry? (ii) Given a dynamical symmetry $\xi, \eta_1, \eta_2, \ldots, \eta_p$, under what conditions does there exist a function $f(q, t)$ such that Φ is conserved?

We first point out that a consideration of the second question makes it clear why it is not useful to allow f to depend on $\dot{q}$. For, if it did, then for arbitrary ξ, η_l we could always choose

$$f(q, \dot{q}, t) = -(\xi\dot{q}_l - \eta_l)\, \partial L/\partial \dot{q}_l + \xi L + \Psi(q, \dot{q}, t), \tag{19}$$

where $\Psi(q, \dot{q}, t)$ is any conserved quantity. In this case, (18) becomes

$$\Phi = \Psi(q, \dot{q}, t).$$

Thus, *any* conserved quantity Ψ could be said to be associated with *any* arbitrary transformation defined by ξ, η_l. This degree of generality is clearly not profitable, so that f is always assumed to be of the form $f(q, t)$.

Differentiating (18) totally with respect to time, we find

$$\dot{\Phi} = (\xi\dot{q}_l - \eta_l)\mathscr{F}_l - E\{L\} - \dot{\xi}L + \dot{f}, \tag{20}$$

where

$$\mathscr{F}_l = (\mathrm{d}/\mathrm{d}t)(\partial L/\partial \dot{q}_l) - \partial L/\partial q_l.$$

If Φ is conserved so that $\dot{\Phi} = 0$, and if the Euler equations are satisfied, so that $\mathscr{F}_l = 0$, then

$$E\{L\} + \dot{\xi}L = \dot{f}. \tag{21}$$

Conversely, if an $f(q, t)$ can be found such that (21) is satisfied, then Φ is conserved.

Referring to the condition (12) for the existence of a dynamical symmetry, we note that if $\xi(q, \dot{q}, t)$ is chosen arbitrarily, the equations may be solved for $\eta_1(q, \dot{q}, t) \ldots \eta_p(q, \dot{q}, t)$. Because of this freedom in the choice of $\xi(q, \dot{q}, t)$, we can construct an infinity of independent dynamical symmetries, all of which fail to satisfy (21). Thus, none of these constructed symmetries will determine a Noether quantity (18). Similarly, in (21) we may arbitrarily assign the $p+1$ functions $f(q, t)$, $\eta_l(q, \dot{q}, t)$, and solve the resulting differential equation for $\xi(q, \dot{q}, t)$. Each of these sets ξ, η_l, then determines a conserved quantity Φ; and because the η_l are arbitrary, the sets can be so determined that they do not satisfy (12). We may therefore construct a $p+1$-fold infinity of conserved quantities, each having the property that the associated ξ, η_l do *not* define a dynamical symmetry. We thus see that in the case of dynamical symmetries, the connection between Noether conserved quantities and invariance transformations is not very pronounced. Nevertheless, it is possible to describe a procedure which establishes a fairly direct relationship between dynamical symmetries and constants of the motion. In this approach, the Lagrangian formalism is not utilised, and the particular representation of a conserved quantity in Noether form is not available.

4. Dynamical symmetries and general systems

We begin by noting that the general solution of (3) may be written

$$q_l = q_l(t) \qquad (l = 1, 2, \ldots, p) \tag{22}$$

and depends on $2p$ arbitrary constants A_k. A set of $2p$ equations may be obtained by adjoining to (22) the p equations obtainable by time differentiation; this resulting set determines the $2p$ constants as functions of q_l, $\dot{q}_l$, and t. These functions of q_l, $\dot{q}_l$, and t are of course constant in time, since the A_k are constants; moreover, any combination of these functions yields further conserved quantities. Naturally, this procedure is not of much use in determining conserved quantities, since it requires that the general solution already be known. However, if a dynamical symmetry is known, then the above considerations lead to an approach which can yield a conserved quantity without knowledge of the general solution.

A characteristic property of a symmetry of (3) is that it permutes solutions of (3) among themselves. Any transformation which carries one solution of (3) into another can be considered to have the effect of changing one set of constants $A_1, A_2, \ldots, A_{2p}$ into another set $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{2p}$. In particular, if a one-parameter Lie group permutes solutions among themselves, then there exists a corresponding Lie group acting on the constants $A_1, A_2, \ldots, A_{2p}$. Suppose that this group has an invariant $S(A_1, A_2, \ldots, A_{2p})$; that is, a function of the A_k having the property

$$S(A_1, A_2, \ldots, A_{2p}) = S(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{2p}). \tag{23}$$

From this invariant can be constructed a conserved quantity $I(q, \dot{q}, t)$, by using the above-mentioned expression for the A_k in terms of q_l, $\dot{q}_l$, and t. Specifically, if we replace the constants by their representations in terms of coordinates, velocities, and time, then equation (23) assumes the form

$$I(q, \dot{q}, t) = I(Q, \dot{Q}, T), \tag{24}$$

and thus $I(q, \dot{q}, t)$ is a conserved quantity for (3). It is natural to consider this constant of the motion $I(q, \dot{q}, t)$ as being associated with the given dynamical symmetry. Furthermore, $I(q, \dot{q}, t)$ can be found without knowing the general solution of equation (3), since it is clear from the relationship between $I(q, \dot{q}, t)$ and $S(A_1, A_2, \ldots, A_{2p})$ that $I(q, \dot{q}, t)$ is an invariant of the dynamical symmetry group itself; that is,

$$E\{I\} = 0. \tag{25}$$

To show this analytically we apply the group operation (10) to the invariant $I(q, \dot{q}, t)$ and obtain

$$\exp(\theta E)\, I(q, \dot{q}, t) = I(Q, \dot{Q}, T) = S(\tilde{A}) = S(A) = I(q, \dot{q}, t). \tag{26}$$

From (26) we note that $\exp(\theta E)\, I(q, \dot{q}, t) = I(q, \dot{q}, t)$, from which (25) follows. This result leads to a method whereby a conserved quantity for (3) might be sought when a dynamical symmetry of (3) is known. The procedure requires that the invariants of the symmetry group be found, for instance by solution of equation (25). Among these invariants will be found the conserved quantity $I(q, \dot{q}, t)$.

It should be noted that the considerations leading to (25) may be applied also to point symmetries if we interpret the operator

$$E = \xi(\partial/\partial t) + \eta_l(\partial/\partial q_l) + [\dot{\eta}_l - \dot{\xi}\dot{q}_l)]\, \partial/\partial \dot{q}_l$$

as representing the generator of the first extension of the group

$$G = \xi(\partial/\partial t) + \eta_l(\partial/\partial q_l);$$

here ξ and η_l are independent of $\dot{q}$, and

$$\dot{\xi} = \partial\xi/\partial t + (\partial\xi/\partial q_l)\dot{q}_l \qquad \dot{\eta}_l = \partial\eta_l/\partial t + (\partial\eta_l/\partial q_m)\dot{q}_m.$$

In particular, for Lagrangian systems, we may now associate a conserved quantity with any point symmetry not possessing a Noether-type constant of the motion; that is, with any point symmetry which leaves the equations of motion invariant but not the action (Lutzky 1978). However, the considerations leading to (25) are even more general, since they do not require that the system be representable by a Lagrangian at all. Consequently, we may state the following general theorem for point symmetries:

Let the group generated by $E = \xi(q, t)\, \partial/\partial t + \eta_l(q, t)\, \partial/\partial q_l$ be a point symmetry of a dynamical system. Then a conserved quantity for the system may be associated with this symmetry group, and may be found among the invariants of the first extended group. This generalises a theorem proved previously only for the case of a Noether conserved quantity in a one-dimensional Lagrangian system: for that special case the statement is that the Noether conserved quantity associated with a point symmetry is an invariant of the first extended group (Lutzky 1978).

The fact that (25) holds for dynamical symmetries enables us to state a theorem for dynamical symmetries which has previously been proved only for point symmetries: if a dynamical symmetry changes a given solution into another, then both solutions possess the same value of the conserved quantity associated with the symmetry. This follows from (25) because $E\{I\}$ gives the change in the invariant I (to first order) due to the change from one solution to another brought about by the action of the symmetry operator.

5. Example

To illustrate the preceding development, we consider the one-dimensional harmonic oscillator, whose equation of motion is given by (13). The condition for a dynamical symmetry, given by (15), is satisfied by (16*a*), with group generator (17). The invariants of this dynamical symmetry group are the independent solutions of the equation

$$E\{\phi\} = q\dot{q}(\partial\phi/\partial t) - q^3(\partial\phi/\partial q) - (2q^2 + \dot{q}^2)\dot{q}(\partial\phi/\partial\dot{q}) = 0 \tag{27}$$

and may be written

$$\phi_1 = \frac{q}{\dot{q}}\sqrt{q^2+\dot{q}^2} \qquad \phi_2 = \frac{q\cos t - \dot{q}\sin t}{\dot{q}\cos t + q\sin t}. \tag{28a, b}$$

It can easily be verified that ϕ_2 is a conserved quantity for the harmonic oscillator, since $\dot{\phi}_2 = 0$ if (13) is satisfied.

Since the exact solution is known for this problem, we may explicitly verify the connection between the symmetry group generated by (27) and the corresponding group acting on the constants of integration.

The finite transformations generated by (27) may be given in the form

$$Q = q/(1+2\theta q^2)^{1/2} \tag{29a}$$

$$\dot{Q} = \dot{q}/\{(1+2\theta q^2)[1+2\theta(q^2+\dot{q}^2)]\}^{1/2} \tag{29b}$$

$$T = t + \tan^{-1}\{(q/\dot{q})[1+2\theta(q^2+\dot{q}^2)]^{1/2}\} - \tan^{-1}(q/\dot{q}). \tag{29c}$$

Let the general solution of (13) be given in the form

$$q = A\cos t + B\sin t, \tag{30a}$$

with derivative

$$\dot{q} = -A\sin t + B\cos t. \tag{30b}$$

Using (30) in (29*a*) and (29*c*) and eliminating t from the resulting two equations yields the transformed curve in Q, T space:

$$Q = \tilde{A}\cos T + \tilde{B}\sin T, \tag{31}$$

where

$$\tilde{A} = A/[1+2\theta(A^2+B^2)]^{1/2} \qquad \tilde{B} = B/[1+2\theta(A^2+B^2)]^{1/2}. \tag{32a, b}$$

Equations (32) represent the group transformation induced on A, B by the symmetry transformation (29) acting on solution curves of (13). Note that an invariant of this transformation is A/B, since, clearly,

$$\tilde{A}/\tilde{B} = A/B.$$

Solving for A and B from (30) we find

$$\frac{A}{B} = \frac{q\cos t - \dot{q}\sin t}{\dot{q}\cos t + q\sin t},$$

which equals ϕ_2 in (28*b*). We thus have an explicit demonstration of the way the induced transformation on the integration constants leads to a conserved quantity associated with the dynamical symmetry; this conserved quantity is, in fact, an invariant of the dynamical symmetry.

We note that with the harmonic oscillator Lagrangian $L = \dot{q}^2 - q^2$, and with E given by (17), the quantity $E\{L\} + \xi L$ cannot be represented in the form $\partial f/\partial t + \dot{q}\ \partial f/\partial q$, with $f = f(q, t)$; thus the above dynamical symmetry cannot lead to a conserved quantity of Noether form. We conclude by presenting a dynamical symmetry which does lead to a Noether conserved quantity for the harmonic oscillator.

Let $\eta = -\dot{q}$, $\xi = 0$; these clearly satisfy condition (15) and therefore define a symmetry transformation for the harmonic oscillator. The generator is

$$E = -\dot{q}\ \partial/\partial q + q\ \partial/\partial \dot{q}$$

and

$$E\{L\} + \dot{\xi} L = 4q\dot{q}.$$

It is easy to find an $f(q, t)$ such that $\dot{f} = 4q\dot{q}$; we may put

$$f = 2q^2.$$

Then the Noether constant of the motion is

$$\Phi = (\xi\dot{q} - \eta)\ \partial L/\partial \dot{q} - \xi L + f = 2(\dot{q}^2 + q^2),$$

which is the energy for the oscillator. Note also that

$$E\{\dot{q}^2 + q^2\} = 0,$$

as it must according to the general theory.

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